Reasoning under uncertainty

Tuesday, October 22, 2019 2:10 PM

- Agents may not have complete information
  - Environment
    - things are hidden
    - Sensors are not fully reliable.
  - Effects of actions
    - rolling ~~the~~ dice
    - drawing a card
    - unreliable actions, - noise on actions
- Make informed Actions.

• Probability

- The Calculus of Gambling
- The Calculus of Belief
  - Thomas Bayes.
  - Subjective Probability
    "of the subject"

"Chevalier de Mer"
≡ Bon Vivant ≡

Pierre de Fermat.
Blaise Pascal.

• Worlds in terms of variables.

- fluents / algebraic Variables /
- variables have domain sets of values they can take.

S dom(S) = {square, circle, star}

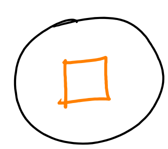
C dom(C) = {blue, orange}

F dom(F) = {true, False}

If domain is finite : is called discrete

- a world is an assignment of values to variables:

S = star
C = blue
F = true



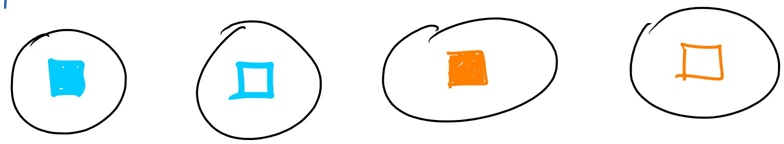
S = square
C = orange
F = false

If domain of all variables is descrete, then there is a finite number of words.
(we will limit ourselves to finite possible worlds)

• A primitive proposition is a boolean expression built from:
assignments of values to variables

- A primitive proposition is a boolean expression built from:
  - assignments of values to variables
  - relational operators $<$ $=$ $>$ $\leq$ $\geq$

C = square



- A probability measure P is a function from worlds to $\mathbb{R}^+$ such that.

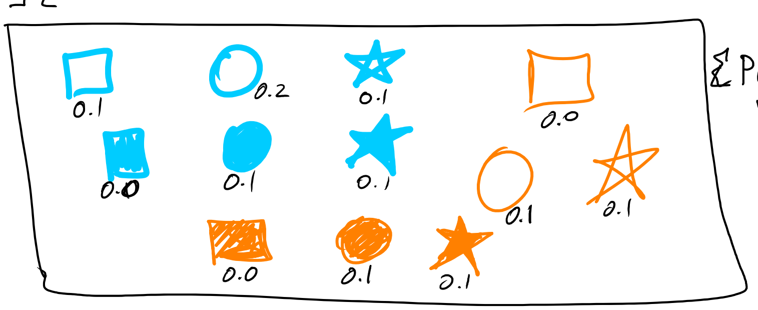
$$\sum_{w \in \Omega} P(w) = 1.0 \qquad \Omega \text{ set of all worlds.}$$

the probability P of a proposition. α

$$P(\alpha) = \sum_{w:\, \alpha \text{ is true in } w} P(w)$$

Example

Ω



$$\sum_{w \in \Omega} P(w) = 1.0$$

$P(S = \text{circle}) = 0.5$ $\qquad$ $P(C = \text{Orange}) = 0.4$

$P(S = \text{star}) = 0.4$ $\qquad$ $P(C = \text{Blue} \land \neg(S = \text{Star})) = 0.4$

- Probability distribution of a variable X: $P(X)$ is a function from $\text{dom}(X)$ to Real values such that $P(x)$ where $x \in \text{dom}(X)$ is equal to $P(X = x)$.

$$P(C) = \begin{array}{l} P(C = \text{blue}) = 0.6 \\ P(C = \text{orange}) = 0.4 \end{array}$$

P(X)	
0.6	blue
0.4	orange

- for a set of variables X, Y, $P(X, Y)$ is a function from every $x \in \text{dom}(X)$ and $y \in Y$ to real values. $P(x, y) = P(X = x \land Y = y)$

every $x \in dom(X)$ and $y \in Y$ to real values. $P(x,y) = P(X=x \wedge Y=y)$

$P(C, F)$

	$P(C,F)$
$P(C=blue \wedge F=T)$	0.2
$P(C=blue \wedge F=F)$	0.4
$P(C=orange \wedge F=T)$	0.2
$P(C=orange \wedge F=F)$	0.2

- if $X_0, X_1, \ldots X_n$ are all your Variables $P(X_0, X_1, X_2 \ldots, X_n)$ is called joint probability distribution

- $0 \leq P(\alpha)$ the probability / belief on a proposition α is never negative.
- $P(\top) = 1$ when $\top$ is a tautology 'C = blue $\vee$ C = orange'
- $P(\alpha \vee \beta) = P(\alpha) + P(\beta)$ as long as α and β are mutually exclusive
  - there is no world in which both α and β are both true.

  'S = Square' (α) 'S = star' (β) $P(\alpha \vee \beta) = P(\alpha) + P(\beta)$
- $P(\neg\alpha) = 1 - P(\alpha)$

- Reasoning by cases

$$P(\alpha) = P(\alpha \wedge \beta) + P(\alpha \wedge \neg\beta)$$

$$P(S = star) = P(S=star \wedge C=blue) + P(S=star \wedge C=orange)$$

$$P(\alpha) = \sum_{\text{for } d \in dom(V)} P(\alpha \wedge V=d)$$

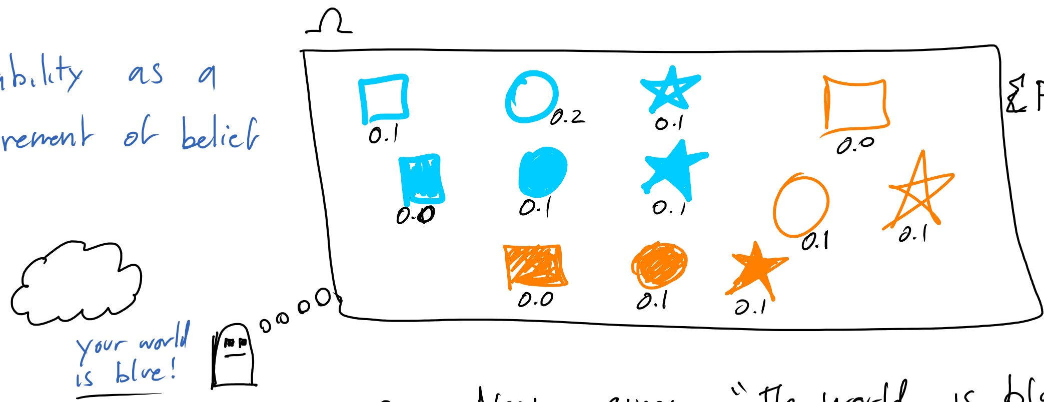
- for non mutually exclusive propositions

$$P(\alpha \vee \beta) = P(\alpha) + P(\beta) - P(\alpha \wedge \beta)$$

Conditional Probability:

Ω

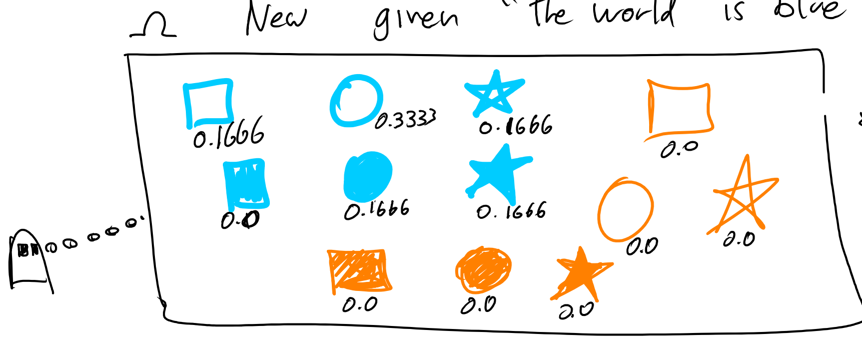
Probability as a measurement of belief



$$\sum_{w \in \Omega} P(w) = 1.0$$

$p(\text{Color} = \text{Blue}) = 0.6.$

New given "The world is blue"



$$\sum p(w|e) = 1$$

$p(h|e)$: conditional probability of h given e

h = proposition

e = proposition

Derive the new probability measure:

$$1 = \sum p(w|e)$$

$$= \sum_{w:\ e \text{ is true}} p(w|e) + \sum_{w:\ e \text{ is false.}} p(w|e)$$

$$p(e) = \sum_{e \text{ is true in } w} p(w)$$

$$1 = c \cdot p(e) + \emptyset$$

$$\underline{c = \frac{1}{p(e)}}$$

Derive the prob of a statement α given e

$$p(\alpha|e) = \sum_{\alpha \text{ is true in } w} p(w|e)$$

$$= \sum_{\alpha \wedge e \text{ is true}} p(w|e) + \sum_{\alpha \wedge \neg e \text{ is true}} p(w|e)$$

$$= \sum_{\alpha \wedge e \text{ are true.}} \frac{1}{p(e)} \cdot P(w) + \emptyset$$

$$p(\alpha|e) = \frac{p(\alpha \wedge e)}{}$$

$\alpha \wedge e$ are true.

$$p(\alpha|e) = \frac{p(\alpha \wedge e)_3}{p(e)}$$

Example

Flu	Sneeze	Snore	P
T	T	T	0.064
T	T	F	0.096
T	F	T	0.016
T	F	F	0.024
F	T	T	0.096
F	T	F	0.144
F	F	T	0.224
F	F	F	0.336

- $p(flu \wedge sneeze) = 0.16$
- $p(flu \wedge \neg sneeze) = 0.04$
- $P(flu) = 0.20$
- $P(sneeze \mid flu) = \frac{p(sneeze \wedge flu)}{p(flu)}$
- $p(\neg flu \wedge sneeze) = 0.24$
- $p(flu \mid sneeze) = \frac{p(flu \wedge sneeze)}{p(sneeze)}$

$$= \frac{0.16}{0.4} = 0.4$$

$$p(h|e) = \frac{p(h \wedge e)}{p(e)}$$

The Chain Rule

$$p(h \wedge e) = p(h|e) \cdot p(e)$$

$$\begin{aligned} p(a \wedge b \wedge c) &= p(a|b \wedge c) \cdot p(b \wedge c) \\ &= p(a|b \wedge c) \cdot p(b|c) \cdot p(c) \end{aligned}$$

$$p(a_1 \wedge a_2 \wedge a_3 \wedge \dots \wedge a_n) = p(a_1|a_2 \wedge a_3 \dots a_n) \cdot$$
$$p(a_2|a_3 \wedge a_4 \dots a_n) \cdot$$
$$p(a_3|a_4 \wedge a_5 \dots a_n) \cdot$$
$$\vdots$$
$$p(a_{n-1}|a_n) \cdot$$
$$p(a_n)$$

$$= \prod_{i=1}^{n} p(a_i \mid a_{i+1} \wedge \dots \wedge a_n)$$

The Bayes Rule:

equal

$$p(a \mid b) = \frac{p(a \wedge b)}{p(b)} \qquad p(b \mid a) = \frac{p(b \wedge a)}{p(a)}$$

$$p(a \mid b) \cdot p(b) = p(b \mid a) \cdot p(a)$$

$$p(a \mid b) = \frac{p(b \mid a) \cdot p(a)}{p(b)}$$ Bayes theorem.

get $p(\text{disease} \mid \text{symtom})$ from $p(\text{symtom} \mid \text{desease})$

$p(\text{fire} \mid \text{alarm})$ from $p(\text{alarm} \mid \text{fire})$.

Excercise:

two kinds of birds — Orange 85%, Blue 15%

one bird has stollen a cookie!!

a witness claims the culprit is Blue.

- Testing the witness correctly identifies birds 80% of the time.

What is the probability that culprit is Blue?

$$P(b) = 0.15 \qquad P(\neg b) = 0.85$$

$$p(b \mid W_b) = \frac{p(W_b \mid b) \cdot P(b)}{p(W_b)} \qquad \begin{array}{l} P(W_b \mid b) = 0.80 \\ P(b) = 0.15 \end{array}$$

$$p(W_b) = p(W_b \wedge b) + p(W_b \wedge \neg b)$$ by reasoning by cases.

$$= p(W_b \mid b) \cdot p(b) + p(W_b \mid \neg b) \cdot p(\neg b)$$

$$0.80 \cdot 0.15 + 0.20 \cdot 0.85$$

$$= 0.29$$

$$p(b \mid W_b) = \frac{(0.80 \cdot 0.15)}{0.29} \approx 0.4138$$

[ref: "thinking fast and slow" by D. Kahneman]

Expected Value:

Let $f(w)$ be a numeric function over <u>worlds</u>

Given a probability measure P the "Expected Value" of $f()$ on P is

$$E_p(f) = \sum_{w \in \Omega} f(w) * P(w)$$

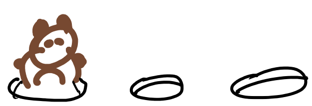
In the special case in which $f(w)$ is boolean $f(w) \begin{cases} 1 & \text{if } a \text{ is true in } w \\ 0 & \text{if } a \text{ is false in } w \end{cases}$

$$E_p(f) = P(a)$$

Conditional Expected Value

$$E_p(f \mid e) = \sum_{w \in \Omega} f(w) \cdot P(w \mid e)$$

Example :



3 holes.
70% that a gopher lives in the hole.

How many Gophers do you expect to find.?

hole #1	#2	#3	P	f
0	0	0	0.027	0
0	0	1	0.063	1
0	1	0	0.063	1
0	1	1	0.147	2
1	0	0	0.063	1
1	0	1	0.147	2
1	1	0	0.147	2
1	1	1	0.343	3

$$p(\#1 = 1) = 0.7$$

f P how many world

$$E_p(f) = 0 \cdot 0.027 + 1 \cdot 0.063 \cdot 3 + 2 \cdot 0.147 \cdot 3 + 3 \cdot 0.343 \cdot 1 = 2.01$$

Example :

U domain [1, 2, ..., 6] $p(w) = \frac{1}{36}$

Example:

d1 domain [1, 2, ..., 6]

d2

$p(w) = \frac{1}{36}$

$f(w) = d1 + d2$

$$E_p(f) = \begin{array}{l} 2 \cdot 1/36 \ + \\ 3 \cdot 2/36 \ + \\ 4 \cdot 3/36 \ + \\ 5 \cdot \\ 6 \cdot \\ 7 \cdot 6/36 \ + \\ \vdots \\ 12 \cdot 1/36 \end{array} = 7$$